# Crosspoint Complexity of Sparse Crossbar Concentrators

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Abstract— A sparse crossbar (n, m, c)-concentrator is a bipartite graph with n inputs and m outputs in which any c or fewer inputs can be matched with an equal number of outputs, where c is called its capacity. We present a number of new results on the crosspoint complexity of such concentrators. First, we describe a sparse crossbar (n, m, m)-concentrator whose crosspoint complexity matches Nakamura–Masson's lower bound for any given n and m. Second, we present a sparse crossbar (2m, m, m)-concentrator with crosspoint complexity also matching Nakamura–Masson's lower bound, and with fixed fan-in and nearly fixed fan-out. Third, we derive an easily computable lower bound on the crosspoint complexity of sparse crossbar (n, m, c)-concentrators. Finally, we show that this bound is attainable within a factor of two when  $n - m \leq c \leq \lfloor m/c \rfloor$ .

Index Terms—Bipartite graph, concentrator, sparse crossbar, crosspoint complexity.

## I. INTRODUCTION

**IVEN AN**  $m \times n$  binary matrix, suppose we wish to **T** determine if, for every c columns of this matrix, there exist c rows such that the intersection of every column in the given set of c columns with a distinct row among those rows contains a "1" entry. This interesting problem about binary matrices can also be posed as a matching question in bipartite graphs [3], and is closely related to the behavior of a switching device, called a sparse crossbar concentrator as follows. The columns in an  $m \times n$  binary matrix represent the inputs of such a device, the rows represent its outputs, and the "1" entries correspond to contacts or crosspoints between the inputs and outputs. The condition that the intersections of every ccolumns with some c rows contain "1" entries characterizes the concentrator's ability to connect any c of its inputs to some c of its outputs. Any sparse crossbar whose columns meet this property will be called a sparse crossbar (n, m, c)concentrator, where c is called its capacity. Sparse crossbar concentrators were introduced by Pinsker [12] who used them to show that full-capacity concentrators (c = m)-concentrators can be constructed with O(n) crosspoints. Full-capacity concentrators play a central role in subscriber loops to multiplex low-rate channels onto higher speed transmission trunks or remote carriers [13]. They are also used in the construction of

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more powerful connectors such as permutation networks and generalized connectors [4], [6], [8]. Bounded capacity sparse crossbars (c < m) can also serve as concentrators when the maximum number of matched inputs is strictly less than the number of outputs, as may be the case in some switching applications.

The problems we consider in this paper deal with proving that certain sparse crossbars exhibit a concentrator behavior and with computing the minimum number of crosspoints it takes to construct a sparse crossbar concentrator. It is desirable that the sparse crossbar concentrators we consider have as few crosspoints as possible. A secondary objective is to keep the fan-out of the inputs and fan-in of the outputs as small as possible and nearly constant over the entire set of inputs and outputs.

A number of results have been reported on the crosspoint complexity of full capacity concentrators. Pinsker proved that there exists an (n, m, m)-concentrator (henceforth called an (n, m)-concentrator) with at most 29n crosspoints [12]. Bassalygo subsequently reduced this bound to 20n crosspoints [2]. Explicit constructions of (n, m, c)-concentrators with O(n)crosspoints were given by Margulis [9] and others [1], [5] for any c, 1 < c < m. While these constructions rely on O(n) crosspoints, they exact  $O(\log n)$  delay. In another direction, Masson [10] and Nakamura and Masson [11] studied the crosspoint complexity of sparse crossbar concentrators. They derived lower bounds on the number of crosspoints in sparse crossbar concentrators and showed that, in certain cases, these bounds are tight. While their bound for full capacity concentrators is easy to compute, to determine their lower bound for bounded capacity concentrators, one must solve a polynomial whose degree depends on the capacity of the concentrator in question.

These results are extended here as follows. First, we describe an (n, m)-concentrator construction, called a fat-andslim crossbar, whose crosspoint complexity matches Nakamura-Masson's lower bound for any given n and m, thereby removing the restriction on the choices of number of inputs and outputs imposed by Masson's binomial network. Second, we present a (2m, m)-concentrator whose crosspoint complexity also matches the same lower bound but with nearly half the fan-out of the first construction. Third, we derive a new lower bound on the crosspoint complexity of sparse crossbar (n, m, c)-concentrators. This bound closely follows Nakamura-Masson's lower bound, but unlike that bound, it is very easy to compute. Finally, we describe a sparse crossbar (n, m, c)-concentrator whose crosspoint complexity

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resides within a factor of two of this new lower bound when  $n - m \le c \le \lfloor m/c \rfloor$ .

# **II. PROBLEM FORMULATION AND APPROACH**

Formally, an (n, m, c)-concentrator is a directed acyclic graph  $\mathcal{G} = (I, O, E)$  with a set of n inputs (I), a set of  $m \leq n$  outputs (O), and a set of edges (E) such that there exist edge-disjoint paths between any c or fewer inputs and an equal number of outputs. The edges in E are called the *crosspoints* of  $\mathcal{G}$ .

A graph  $\mathcal{G} = (I, O, E)$  is called a sparse crossbar if each edge in E lies directly between an input in I and an output in O. A sparse crossbar is called a binomial  $\binom{m}{v}, m$ -network if every one of its  $\binom{m}{v}$  inputs is connected to a distinct subset of v outputs among its m outputs. The number of outputs (inputs) to which an input (output) is connected is called its fan-out (fan-in), and the maximum number of outputs (inputs) to which an input (output) in  $\mathcal{G}$  is connected is called the fan-out (fan-in) of  $\mathcal{G}$ .

To prove that a sparse crossbar  $\mathcal{G} = (I, O, E)$  is an (n, m, c)-concentrator, we will need to show that there exists a *matching* between every c inputs in I and some c outputs in O. That is, for every  $x_1, x_2, \dots, x_c \in I$ , we must show that there exist  $y_1, y_2, \dots, y_c \in O$  such that  $(x_1, y_1), (x_2, y_2), \dots, (x_c, y_c)$  constitute crosspoints in E. In this connection, the following well-known theorem due to Hall will prove invaluable [7].

Theorem 1 (P. Hall): Let O be a finite set and let  $Y_1, Y_2, \dots, Y_r$  be arbitrary subsets of O. There exist distinct elements  $y_i \in Y_i, 1 \le i \le r$  if and only if the union of any k of  $Y_1, Y_2, \dots, Y_r$  contains at least k elements.

Hall's theorem will be invoked in our proofs as follows: To begin with, the set O in the theorem will denote the set of outputs of a sparse crossbar,  $\mathcal{G} = (I, O, E)$ , and  $Y_1, Y_2, \dots, Y_r$ will represent the subsets of all outputs in O which are connected to some r inputs  $x_1, x_2, \dots, x_r$  in I, in that order, i.e.,  $Y_1$  contains all the outputs connected to input  $x_1$ ,  $Y_2$ contains all the outputs connected to input  $x_2$ , and so on. The outputs in  $Y_i$  are called the neighbors of  $x_i$ , and  $Y_i$  is called the neighbor set of  $x_i, 1 \leq i \leq r$ . With this setup, we will attempt to establish that the union of  $Y_1, Y_2, \dots, Y_r$  contains at least r outputs for any choices of  $x_1, x_2, \dots, x_r$  in I, and any  $r, 1 \leq r \leq c$ , where c is the projected capacity of  $\mathcal{G}$ . This fact will then be used together with Hall's theorem to conclude that  $\mathcal{G}$  is an (n, m, c)-concentrator.

As for computing lower bounds on the crosspoint complexity of sparse crosbar concentrators, our proofs will rely on some elementary observations concerning the minimum number of neighbors of every subset of outputs must have in such graphs. This minimum will then be coupled with the required capacity of the sparse crossbar in question to obtain a lower bound on its crosspoint complexity.

# **III. FULL-CAPACITY CONCENTRATORS**

Nakamura and Masson [11] derived a lower bound on the crosspoint complexity of full-capacity sparse crossbar concentrators, and proved that this bound is tight for a binomial network. Here we describe a new full-capacity sparse crossbar concentrator whose crosspoint complexity matches the same lower bound for any number of inputs and any number of outputs. In contrast, the ratio of the number of outputs to the number of inputs of a binomial  $\binom{m}{m-2}, m$ -network approaches 0 as  $m \to \infty$ . Another problem with the binomial  $\binom{m}{m-2}, m$ -network is that its fan-out (which is m-2) is nearly as large as its number of its outputs, and its fan-in (which is (m-1)(m-2)/2) is also nearly as large as its number of inputs. We describe a sparse crossbar (2m, m)-concentrator construction that is optimal with respect to its crosspoint complexity, and has fan-out which is almost half the number of its outputs and fan-in which is almost half of its inputs.

We first recall the following lower bound from [11].

Theorem 2 (Nakamura–Masson): Any sparse crossbar (n, m)-concentrator requires at least m(n - m + 1) crosspoints.

Definition 1: Let  $\mathcal{G} = (I, O, E)$  be a sparse crossbar with n inputs and m outputs. Suppose that I is partitioned into two sets,  $I_1$  and  $I_2$ , where  $|I_1| = n - m$  and  $|I_2| = m$ .  $\mathcal{G}$  is called an (n, m)-fat-and-slim crossbar if each of the n - m inputs in  $I_1$  is connected to all the m outputs, and if each of the m inputs in  $I_2$  is connected to a single but distinct output.

Theorem 3: For any positive integers m and  $n, m \le n$ , every (n, m)-fat-and-slim crossbar yields an (n, m)-concentrator with a minimum number of crosspoints.

**Proof:** Let  $\mathcal{G} = (I, O, E)$  be an (n, m)-fat-and-slim crossbar, and  $X = \{x_1, x_2, \cdots, x_r\}$  be an arbitrary r-subset of I, where  $1 \leq r \leq m$ , and let  $Y_i$  be the neighbor set of input  $x_i, 1 \leq i \leq r$ . By the construction of  $\mathcal{G}$ , it is obvious that if at least one of  $x_1, x_2, \cdots x_r$  belongs to  $I_1$  then  $Y_1 \cup Y_2 \cup \cdots \cup Y_r$  contains at least m outputs. On the other hand, if all of  $x_1, x_2, \cdots, x_r$  belong to  $I_2$  then  $Y_1 \cup Y_2 \cup \cdots \cup Y_r$ contains exactly r outputs. The concentrator property of  $\mathcal{G}$  thus follows from Hall's theorem. That the crosspoint complexity of  $\mathcal{G}$  matches the lower bound in Theorem 2 is obvious from its construction.

Fig. 1 shows a fat-and-slim crossbar<sup>1</sup> for n = 13 and m = 4. In this particular fat-and-slim crossbar construction, the inputs in  $I_2$  are connected to the four outputs in a diagonal fashion. This is one of 4! possible constructions that can be obtained by permuting the inputs in  $I_2$  onto the 4 outputs in 4! ways. In general, there are  $\binom{n}{m}m! = n!/(n-m)!$  fat-and-slim crossbars with n inputs and m outputs.

Our second construction provides a minimum crosspoint complexity sparse crossbar concentrator with nearly fixed fanout and fixed fan-in when n = 2m.

Theorem 4: Let  $\mathcal{G} = (I, O, E)$  be a sparse crossbar with 2m inputs and m outputs. Let  $O = \{1, 2, 3, \dots, m\}$ , and suppose that I is partitioned into two sets  $I_1 = \{1, 2, 3, \dots, m\}$  and  $I_2 = \{m + 1, m + 2, \dots, 2m\}$ , where  $|I_1| = |I_2| = m$ . Suppose each input in  $I_1$  is connected to all the odd outputs, and also input 2i is connected to output  $2i, 1 \leq i \leq \lfloor m/2 \rfloor$ .

<sup>&</sup>lt;sup>1</sup>The name fat-and-slim crossbar for this construction is coined not so much to capture its topology (i.e., its fat and slim sections), but rather to point out that while the fat-and-slim crossbar has the illusion of having too many crosspoints, in reality, its crosspoint complexity matches the lower bound in Theorem 2.

Concentrator	Crosspoint Comp.	Fanin	Fanout	I/O Restriction
Binomial network	m(n-m+1)	n-m+1	m-2	Yes
Fat-and-Slim Crossbar	m(n-m+1)	n-m+1	m	No
Sparse Crossbar (Theorem 4)	m(n-m+1)	n-m+1	(m/2)+1	Yes

 TABLE I

 COMPARISON OF THREE SPARSE CROSSBAR FULL-CAPACITY CONCENTRATORS



Fig. 1. A (13, 4)-fat-and-slim crossbar.

Likewise, suppose each input in  $I_2$  is connected to all the even outputs, and also input  $2i+1+(m \mod 2)$  is connected to output  $2(i-\lfloor m/2 \rfloor)+1, \lfloor m/2 \rfloor \leq i \leq m-1$  (see Fig. 2).  $\mathcal{G}$  is a (2m,m)-concentrator with a minimum number of crosspoints, with fan-out =  $\lfloor m/2 \rfloor + 1$  and fan-in = m + 1.

*Proof:* The proof of this theorem is similar to the proof of the previous theorem, but requires a more careful inspection of the neighbors of the inputs of  $\mathcal{G}$ . Let X be an arbitrary rsubset of inputs, where  $1 \leq r \leq m$ , and let  $X_1 = X \cap I_1 =$  $\{x_1, x_2, \cdots, x_p\}$  and  $X_2 = X \cap I_2 = \{x'_1, x'_2, \cdots, x'_q\}$ , where r = p + q. It is obvious that if  $X_1 = \emptyset$  or  $X_2 = \emptyset$  then the inputs in X must have at least r neighbors. Otherwise, let  $Y_i$  be the neighbor set of input  $x_i, 1 \le i \le p$ , and let  $Y_i$ be the neighbor set of input  $x'_i, 1 \leq i \leq q$ . It is easy to see that  $Y_1 \cup Y_2 \cup \cdots \cup Y_p$  contains at least  $\lceil m/2 \rceil + \epsilon_1$  outputs, where the first term in the sum accounts for the odd-numbered neighbors of the inputs in  $X \cap I_1$ , and  $\epsilon_1 \geq 0$  accounts for the even-numbered neighbors of the same inputs. Likewise,  $Y'_1 \cup Y'_2 \cup \cdots \cup Y'_a$  contains at least  $|m/2| + \epsilon_2$  outputs, where the first term in the sum accounts for the even-numbered neighbors of the inputs in  $X \cap I_2$ , and  $\epsilon_2 \ge 0$  accounts for the odd-numbered neighbors of the same inputs. Now, let

and

$$Y' = Y_1' \cup Y_2' \cup \dots \cup Y_a'$$

 $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_p$ 

Then the number of neighbors of the inputs in X is given by

$$|Y| + |Y'| - |Y \cap Y'|.$$

Furthermore, the indices of the outputs in Y and Y' show that the intersection of the two sets contains no more than

$$\min\{\epsilon_1, m/2\} + \min\{\epsilon_2, m/2\} = \epsilon_1 + \epsilon_2$$



Fig. 2. A (12, 6)-concentrator with a minimum number of crosspoints.

outputs. Therefore, the inputs in X must have at least

$$[m/2] + \epsilon_1 + |m/2| + \epsilon_2 - (\epsilon_1 + \epsilon_2) = m \ge r$$

neighbors. Hence, by Hall's theorem,  $\mathcal{G}$  is an (n,m)concentrator. Furthermore, the number of crosspoints in this
construction is given by (m+1)m = (2m - m + 1)m, and
this matches the lower bound of Theorem 2 with n = 2m.
The fan-in and fan-out expressions are obvious.

The key features of the two concentrator constructions described in this section along with the binomial network are summarized in Table I. All three constructions are optimal with respect to their crosspoint complexity. The advantage of the fat-and-slim crossbar over the other two crossbars is the fact that it does not place any restrictions on its number of inputs and outputs, whereas the advantage of the sparse crossbar described in Theorem 4 is its relatively small fan-out.

## IV. BOUNDED-CAPACITY CONCENTRATORS

The lower bound stated in Theorem 2 applies only to full-capacity sparse crossbar concentrators. In this section, we consider the extension of this result to bounded-capacity concentrators.

# A. Nakamura-Masson's Lower Bound

We first recall the lower bound established in [11]. *Theorem 5 (Nakamura-Masson):* Any sparse crossbar (n, m, c)-concentrator requires nx crosspoints where x satisfies

$$\frac{\binom{c}{x}}{\binom{m}{x}} n(c-x) - c^2 + c = 0.$$
(1)

For some values of n, m, and c, this bound is tight. In particular, the following holds.

Corollary 1 (Nakamura-Masson): For all  $m, v \ge 2$ , the binomial  $\binom{m}{v}, m$  network is a sparse crossbar  $\binom{m}{v}, m, v + 2$ )-concentrator with a minimum number of crosspoints.

*Remark 1:* We note that the Nakamura–Masson's lower bound is also tight for c = 1 and c = m. (c = 1 gives *n* crosspoints and c = m gives m(n - m + 1) crosspoints both of which are clearly upper bounds as well, the latter by Theorem 3.)

Except these three cases, it is not known if Nakamura-Masson's lower bound is also tight for other bounded capacity concentrators. Part of the difficulty stems from the implicit nature of the lower bound since one must solve for x in (1) before the lower bound can be determined. When simplified, the expression on the left-hand side gives a polynomial of degree m - c + 1 in x which is cumbersome to solve for an exact value of x especially if  $m \gg c$ . Instead, Nakamura and Masson worked out a lower bound on x which is given by

$$x \ge c - \binom{m}{c} \frac{c(c-1)}{n}.$$
 (2)

Combining this lower bound on x with Theorem 5 shows that any (n, m, c)-concentrator requires

$$nx \ge nc - \binom{m}{c} c(c-1) \tag{3}$$

crosspoints. If m and c are fixed, this lower bound is asymptotically equivalent to nc as  $n \to \infty$ .

# B. The New Lower Bound

The asymptotic lower bound given in (3) is useful when  $n \gg m$ , but, in most cases, one would be more interested in (n, m)-concentrators, where m scales with n. In this case, this lower bound is negative for all  $c \ge (n/m) + 1$ , and  $m \gg c$ . This can be seen by noting that  $\binom{m}{c} \ge m$ , for  $m \gg c$ , and hence

$$nc - \binom{m}{c}c(c-1) \le c(n-m(c-1)) \le 0$$

for all  $c \ge (n/m) + 1$ .

Our next result gives a new lower bound on the crosspoint complexity of (n, m, c)-concentrators which is very easy to compute and closely follows Nakamura–Masson's exact lower bound for all m = O(n).

Theorem 6: Any sparse crossbar (n, m, c)-concentrator requires  $\lceil m(n-c+1)/m-c+1 \rceil$  crosspoints.

**Proof:** Let  $\mathcal{G} = (I, O, E)$  be a sparse crossbar (n, m, c)concentrator. Then each (m - c + 1)-subset of outputs in Oshould be connected to at least n - c + 1 inputs in I, since, otherwise, there will exist some c inputs that are connected only to c - 1 outputs,<sup>2</sup> contradicting the fact that  $\mathcal{G}$  is an (n, m, c)-concentrator. Let  $P_{m-c+1}(O)$  denote the collection of all (m - c + 1)-subsets of O,  $d_i$  denote the fan-in of output  $i, 1 \le i \le m$ , and  $\rho_i$  denote the number of (m - c + 1)-subsets of O that contain output i. Since the number of neighbors

 $^{2}$ A subset of outputs Y is said to be connected to a subset of inputs X if there exists a crosspoint between every input in X and some output in Y.

(inputs) of the outputs in an (m - c + 1)-subset of O cannot be larger than the sum of the crosspoints connected to the outputs in that subset, we have

$$\sum_{Y \in P_{m-c+1}(O)} \sum_{i \in Y} d_i \ge \binom{m}{m-c+1} (n-c+1) \quad (4)$$

or, equivalently,

$$\sum_{i=1}^{m} \rho_i d_i \ge \binom{m}{m-c+1} (n-c+1) \tag{5}$$

where the expressions on the left-hand side of both inequalities sum the in-degrees of the outputs in O over all of its (m - c + 1)-subsets. Now, let  $\kappa_{\mathcal{G}}(n, m, c)$  denote the number of crosspoints in  $\mathcal{G}$ . Noting that  $\rho_i = \binom{m-1}{m-c}$  and

$$\kappa_{\mathcal{G}}(n,m,c) = \sum_{i=1}^{m} d_i$$

(5) gives

$$\binom{m-1}{m-c}\kappa_{\mathcal{G}}(n,m,c) \ge \binom{m}{m-c+1}(n-c+1) \quad (6)$$

or

$$\kappa_{\mathcal{G}}(n,m,c) \ge \left[\frac{\binom{m}{m-c+1}(n-c+1)}{\binom{m-1}{m-c}}\right]$$
(7)

which reduces to

$$\kappa_{\mathcal{G}}(n,m,c) \ge \lceil m(n-c+1)/m-c+1 \rceil$$

when simplified.

We note that, as with Nakamura-Masson's exact lower bound, this new lower bound reduces to the lower bound given in Theorem 3 when c = m, and to n when c = 1.

Fig. 3 shows how the two bounds are related together for various other values of c. It should be pointed out that the new lower bound gets closer to Nakamura-Masson's lower bound as  $n/m \rightarrow 1$ . We also note that computing the exact values of Nakamura-Masson lower bound is much more time-consuming than computing new lower bound as the former requires solving a polynomial of degree m - c + 1.

# C. An Almost Tight Construction

At this point, it is reasonable to ask whether we can construct a bounded-capacity sparse crossbar (n, m, c)concentrator for any n, m, and c with a minimum crosspoint complexity. Unlike the full-capacity case, the resolution of this question is complicated by two related facts. First, the new lower bound we derived in the previous section is not as tight as Nakamura-Masson's lower bound. Second, Nakamura-Masson's lower bound is not explicit enough to suggest a bounded capacity sparse crossbar concentrator construction whose crosspoint complexity may somehow match it by a constant factor.



Fig. 3. Comparison of Nakamura-Masson's lower bound with the new lower bound.

Given these facts, we present in this section a boundedcapacity sparse crossbar (n, m, c)-concentrator whose crosspoint complexity comes within a factor of two of the new lower bound, when  $n - m \leq c \leq \lfloor m/c \rfloor$ . A similar construction for other values of n, m, and c remains an open problem.

Theorem 7: Let  $\mathcal{G} = (I, O, E)$  be a sparse crossbar, where  $n - m \leq c \leq \lfloor m/c \rfloor$ . Suppose that the inputs in I are partitioned into two sets  $I_1 = \{1, 2, \dots, n-c\}$ , and  $I_2 = \{n-c+1, n-c+2, \dots, n\}$ , and that the first  $\lfloor m/c \rfloor c$  outputs in O are partitioned into  $\lfloor m/c \rfloor$  sets

$$O_i = \{(i-1)c+1, (i-1)c+2, \dots, ic\}, \quad 1 \le i \le \lfloor m/c \rfloor.$$

Let the inputs in  $I_2$  be connected to the outputs in  $O_1, O_2, \dots, O_{\lfloor m/c \rfloor}$  in a diagonal fashion, i.e., let

$$(n-c+j,(i-1)c+j) \in E, 1 \le i \le \lfloor m/c \rfloor, \quad 1 \le j \le c.$$

Furthermore, let input i in  $I_1$  be connected to output i in  $O, 1 \leq i \leq n - c$ . Then  $\mathcal{G}$  (see Fig. 4) is an (n, m, c)-concentrator with n - c + |m/c|c crosspoints.

*Proof:* We need to show that every subset of c inputs can be matched with some c outputs,  $n - m \le c \le \lfloor m/c \rfloor$ . It is obvious from the construction that if these c inputs all belong to  $I_1$  or all belong to  $I_2$  then this can be done. So,



Fig. 4. The structure of the sparse crossbar construction in Theorem 7. (The thick lines indicate where the crosspoints are located.)

consider an arbitrary but fixed set of  $\alpha \leq c$  inputs X, and define  $X_1 = X \cap I_1 \neq \emptyset$  and  $X_2 = X \cap I_2 \neq \emptyset$ . Let  $\alpha_1 = |X_1|, \alpha_2 = |X_2|$  and let  $Y_1$  and  $Y_2$  be the sets of neighbors of  $X_1$  and  $X_2$  in  $\mathcal{G}$ . We have

$$|Y_1 \cup Y_2| = |Y_1| + |Y_2| - |Y_1 \cap Y_2|$$

and by the construction of  $\mathcal{G}, |Y_1| = \alpha_1$ 

$$|Y_2| = \alpha_2 \lfloor m/c \rfloor, |Y_1 \cap Y_2| \le \min\{\alpha_1, \alpha_2 \lfloor m/c \rfloor\}$$

so that

$$|Y_1 \cup Y_2| \ge \alpha_1 + \alpha_2 \lfloor m/c \rfloor - \min\{\alpha_1, \alpha_2 \lfloor m/c \rfloor\}.$$

Now, since  $\alpha_2 \ge 1$  and  $c \le |m/c|$ , we have

$$\alpha_2\lfloor m/c\rfloor \ge \lfloor m/c\rfloor \ge c \ge \alpha_1.$$

Hence

$$|Y_1 \cup Y_2| \ge \alpha_1 + \alpha_2 \lfloor m/c \rfloor - \alpha_1 = \alpha_2 \lfloor m/c \rfloor \ge c \ge \alpha.$$

Therefore, by Hall's theorem,  $\mathcal{G}$  is an (n, m, c)-concentrator for any  $c, n - m \leq c \leq \lfloor m/c \rfloor$ . Moreover, its construction reveals that it encompasses  $n - c + \lfloor m/c \rfloor c$  crosspoints, where the first term accounts for the number of crosspoints connected to the inputs in  $I_1$ , while the second term accounts for the number of crosspoints connected to the inputs in  $I_2$ .

The crosspoint complexity of this sparse crossbar concentrator is quite close to the lower bound derived in the previous section. To see this, we note that

$$n-c+\lfloor m/c\rfloor c\leq n-c+m\leq (n-c)m/(m-c+1)+m$$

so that the crosspoint complexity of this sparse crossbar concentrator is within a factor of two of the lower bound derived in the previous section. We also note that a more direct construction of a sparse crossbar (n, m, c)-concentrator obtained by connecting each input to some c outputs yields a crosspoint complexity of nc which gives  $O(n^{1.5})$  crosspoints as compared to O(n) crosspoints of this construction for  $n - m \le c \le \lfloor m/c \rfloor$ .

We illustrate this bounded-capacity sparse crossbar construction in Fig. 5 for n = 12, m = 10, and c = 3. The shaded boxes on the left show how an input in  $I_2$  can be



Fig. 5. Illustration of the sparse crossbar construction in Theorem 7 for n = 12, m = 10, and c = 3.

blocked by as many as c-1 inputs in  $I_1$ , demostrating the necessity of c crosspoints per each input in  $I_2$ . This particular construction uses 12 - 3 + 9 = 18 crosspoints, whereas the lower bound yields

$$[(n-c+1)m/(m-c+1)] = [10 \times 10/8] = 13$$

crosspoints.

#### V. CONCLUDING REMARKS

The three main results of this paper are 1) the construction of a full-capacity sparse crossbar concentrator with a minimum number of crosspoints for any number inputs and any number of outputs, 2) a new, easily computable lower bound on the crosspoint complexities of bounded-capacity sparse crossbar concentrators, and 3) the construction of a bounded-capacity sparse crossbar concentrator with crosspoint complexity which is linear in its number of inputs (and outputs) when the capacity does not exceed the square root of the number of outputs. These results extend Nakamura-Masson's earlier work on sparse crossbar concentrators in a tangible way, while leaving out the construction of an (n, m, c)-concentrator with a crosspoint complexity which is within a constant factor of either of the lower bounds stated in Section IV for arbitrary n, m, and c as an open problem.

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